THE BOCHNER–FLAT CONE OF A CR MANIFOLD

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Abstract: We construct a Kähler structure (which we call a generalised Kähler cone) on an open subset of the cone of a strongly pseudo-convex CR manifold endowed with a 1-parameter family of compatible Sasaki structures. We determine those generalised Kähler cones which are Bochner-flat and we study their local geometry. We prove that any Bochner-flat Kähler manifold of complex dimension bigger than two is locally isomorphic to a generalised Kähler cone.

1. Introduction

The Bochner tensor of a Kähler manifold is the biggest irreducible component of the curvature tensor under the action of the unitary group. In complex dimension two, the Bochner tensor coincides with the anti-self dual Weyl tensor. A Kähler manifold is Bochner-flat if its Bochner tensor vanishes. Bochner-flat Kähler manifolds represent an important class of Kähler manifolds and have been intensively studied: the local geometry of Bochner-flat Kähler manifolds and its interations with Sasaki geometry has been studied, using the Webster's correspondence, in [8]; complete Bochner-flat Kähler structures on simply connected manifolds have been classified in [3]; generalisations of Bochner-flat Kähler manifolds (like weakly Bochner-flat Kähler manifolds and Kähler manifolds with a hamiltonian 2-form) have also been developed (see, for example, [1], [2], [9]).

An important class of Kähler manifolds is represented by the Kähler cones of Sasaki manifolds. Unfortunately, except when the Sasaki manifold is an open subset of the standard CR sphere with its standard metric as the Sasaki metric, the Kähler cones are not Bochner-flat. In this paper we propose an alternative construction, which is a natural generalisation of the Kähler cone construction and which produces, locally, all Bochner-flat Kähler structures of complex dimension bigger than two. More precisely, we consider, on a fixed CR manifold (N, H, I), a 1-parameter family of Sasaki Reeb vector fields $\{T_r, r \in \mathcal{J}\}$ (with $\mathcal{J} \subset \mathbb{R}^{>0}$ an open connected interval). On the cone manifold $N \times \mathcal{J}$ we define an almost complex structure J, which on $H \subset T(N \times \mathcal{J})$ coincides with I and which sends the radial vector field $V = r\frac{\partial}{\partial r}$ to the vector field T, defined by $T(p, r) := T_r(p)$, for any $(p, r) \in N \times \mathcal{J}$. It turns out that J is integrable and that the pair $(\omega := \frac{1}{4} dd^J r^2, J)$ is a Kähler structure on the open subset of $N \times \mathcal{J}$ where $\omega(V, T) > 0$. Such a Kähler structure will be called a generalised Kähler cone and coincides with the usual Kähler cone of a Sasaki manifold when the family of Reeb vector fields is constant. A strong motivation for this construction comes from the fact that the Bryant's family of Bochner-flat Kähler structures (which have been discovered by Robert Bryant in his classification theorem of complete Bochner-flat Kähler structures on simply connected manifolds [3] and have been further studied in [8]) are generalised Kähler cones. Our main result is the following:

Theorem 1. *Any Bochner-flat Kähler manifold of complex dimension bigger than two is locally isomorphic to a generalised Kähler cone.*

The plan of the paper is the following: in Section 2 we review the theory of Kähler and Sasaki manifolds, which will be useful later on in our study of generalised Kähler cones. In Section 3 we determine the generalised Kähler cones which are Bochner-flat and in Section 4 we study their local geometry. This study will readily imply Theorem 1. The last Section is dedicated to examples. We explain how Kähler manifolds with constant holomorphic sectional curvature, weighted projective spaces and Bryant's family of Bochner-flat Kähler structures fit into our formalism of generalised Kähler cones. We also look at Bochner-flat Kähler generalised Kähler cones of order one and at those which are of Tachibana and Liu type [14].

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2. Notations and earlier results

2.1. **The Bochner tensor of a Kähler manifold.** In this section we recall the definition of the Bochner tensor of a Kähler manifold. We use the formalism developed in [1], [9].

Let (V, g, J) be a real vector space together with a complex structure J and a J-invariant positive definite metric g. We shall identify vectors and covectors of V using the metric g. Let $\omega := g(J\cdot, \cdot)$ be the Kähler form. Recall that the space $\mathcal{K}(V)$ of Kähler curvature tensors of (V, g, J), defined as those curvature tensors which annihilate all J-anti-invariant 2-forms on V, decomposes into a g-orthogonal sum

$$\mathcal{K}(V) := c^*_{\mathcal{K}}\left(\mathrm{Sym}^{1,1}(V)\right) \oplus \mathcal{W}(V), \tag{1}$$

where $c^*_{\mathcal{K}} : \mathrm{Sym}^{1,1}(V) \to \mathcal{K}(V)$ is the adjoint of the Ricci contraction

$$c_{\mathcal{K}} : \mathcal{K}(V) \to \mathrm{Sym}^{1,1}(V), \quad c_{\mathcal{K}}(R)(v, w) := \mathrm{trace} R(v, \cdot, w, \cdot), \quad v, w \in V$$

and has the following expression [9]

$$c^*_{\mathcal{K}}(S) = \frac{1}{2}[\frac{S \wedge \mathrm{Id} + (J \circ S) \wedge J}{2} + \omega \otimes S + \beta \otimes J], \tag{2}$$

where $S \in \mathrm{Sym}^{1,1}(V)$ is a symmetric J-invariant endomorphism of V, "Id" is the identity endomorhism, $\beta \in \Lambda^{1,1}(V)$ is the J-invariant 2-form on V, related to S by $\beta(v, w) := g(SJv, w)$, and, for two endomorphisms S and T of V, $S \wedge T$ is the endomorphism of $\Lambda^2(V)$ defined by the formula

$$(S \wedge T)(v, w) := S(v) \wedge T(w) - S(w) \wedge T(v), \quad v, w \in V.$$

According to the decomposition (1), a Kähler curvature tensor $R \in \mathcal{K}(V)$ decomposes into the sum

$$R = c^*_{\mathcal{K}}(S) + W^K,$$

where $W^K \in \mathcal{W}(V)$ is its principal part (or the Bochner tensor of R) and $S \in \mathrm{Sym}^{1,1}(V)$ is a modified Ricci tensor.

Consider now a Kähler manifold (M, g, J). The curvature R^g of the Kähler metric g, is, at every point $p \in M$, a Kähler curvature tensor of the tangent space (T_pM, g_p, J_p). The principal part of R^g is called the Bochner tensor of (M, g, J) and is a section of the symmetric product $\Lambda^{1,1}(M) \odot \Lambda^{1,1}(M)$. The Kähler manifold (M, g, J) is Bochner-flat if its Bochner tensor vanishes.

2.2. **Review of CR and Sasaki manifolds.** Recall that an oriented (strongly pseudo-convex) CR manifold (N, H, I) has a codimension one oriented subbundle H of the tangent bundle TN, called the contact bundle, and a bundle homomorphism $I : H \to H$ with $I^2 = -\mathrm{Id}$, such that, for every smooth sections $X, Y \in \Gamma(H)$, $[IX, IY] - [X, Y]$ is also a section of H and the integrability condition

$$[IX, IY] - [X, Y] = I\left([IX, Y] + [X, IY]\right) \tag{3}$$

is satisfied. Since N and H are oriented, the co-contact line bundle $L := TN/H$ is also oriented, hence trivialisable. A positive section μ of L defines a contact form $\theta := \eta\mu^{-1}$ on M, where $\eta : TN \to L$ is the natural projection and $\mu^{-1} \in \Gamma(L^*)$ is the dual section of μ, i.e. the natural contraction between μ and μ^{-1} is the function on N identically equal to one. The bilinear form $g(X, Y) := \omega(X, IY) := \frac{1}{2}d\theta(X, IY)$ of the bundle H is independent, up to a positive multiplicative function, of the choice of the contact form and is positive definite - the strongly pseudo-convexity condition. The contact form θ determines a Reeb vector field T, uniquely defined by the conditions $\theta(T) = 1$ and $i_T d\theta = 0$. Note that the Reeb vector field preserves the bundle H, i.e. $[T, X] \in \Gamma(H)$ when $X \in \Gamma(H)$ and hence $\mathcal{L}_T(I)$ is a well-defined endomorphism of H. There is also a Riemannian metric g of N associated to θ, which on H is defined above and such that T is of norm one and orthogonal to H. Finally, we need to recall the definition of the Tanaka connection [12] associated to θ. It is the unique connection ∇ on N with the following three properties: (i) it preserves the bundle H; (ii) I, g and T are ∇ parallel; (iii) the torsion T^∇ of ∇ has the following expression:

$$\begin{aligned} T^\nabla(X, Y) &= 2\omega(X, Y) \\ T^\nabla(T, X) &= -\frac{1}{2} I \mathcal{L}_T(I)(X), \end{aligned}$$

for every $X, Y \in \Gamma(H)$. It turns out that on H, ∇ is determined by a Koszul type formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\left(g(Y, Z)\right) + Y\left(g(X, Z)\right) - Z\left(g(X, Y)\right) \\ &\quad + g([X, Y]^H, Z) - g([X, Z]^H, Y) - g([Y, Z]^H, X) \end{aligned}$$

where $X, Y, Z \in \Gamma(H)$ and for a vector field W of N, $W^H := W - \theta(W)T$ is its g-orthogonal projection on the bundle H. The metric g is called Sasaki if, by definition, T is a Killing vector field for the metric g, or, equivalently, if $\mathcal{L}_T(I) = 0$. In this case, the curvature R^∇ of the Tanaka connection on the bundle H is an element of the tensor product $\Lambda^2(N) \otimes \Lambda^{1,1}(H)$ and its restriction to bivectors of H belongs to $\Lambda^{1,1}(H) \odot \Lambda^{1,1}(H)$ and is a Kähler curvature tensor of the complex Riemannian vector bundle (H, g, I). Its Bochner part – called the Chern-Moser tensor [6], [7] of the CR manifold (N, H, I) – is independent of the choice of the

compatible Sasaki structure on (N, H, I). A CR manifold with vanishing Chern-Moser tensor is called flat. The importance of the Chern-Moser tensor comes from the fact that if the CR manifold is flat and of dimension bigger than three, then it is locally isomorphic with a sphere with its standard CR structure [6], [4]. On the other hand, if g is Sasaki, the complex structure I and the metric g of the bundle H descend on the quotient N/T and determine a Kähler structure on this quotient. (In our conventions, the quotient N/T denotes the space of leaves of the foliation generated by T in a sufficiently small open subset of N, so that N/T is a manifold). Moreover, the Bochner tensor of the Kähler manifold N/T becomes identified with the Chern-Moser tensor of the CR manifold (N, H, I) [13], [7]. Since any Kähler manifold can be locally written as a quotient of a Sasaki manifold under its Reeb vector field, by means of a choice of a local primitive of the Kähler form, it follows that a Bochner-flat Kähler manifold of complex dimension $m \geq 2$, is locally isomorphic with the quotient of a standard CR sphere S^{2m+1} under the Reeb vector field of a compatible Sasaki structure.

2.3. **The local type of Bochner-flat Kähler structures.** The local geometry of Bochner-flat Kähler structures, of complex dimension $m \geq 2$, is of four types [3], [5], [8]. This follows from the fact that the compatible Sasaki structures on the CR sphere S^{2m+1} are determined by elements of the Lie algebra $su(m+1, 1)$, and that there are four conjugacy classes in this Lie algebra (elliptic, hyperbolic, 1-step and 2-step parabolic). In order to explain this, it is convenient to identify S^{2m+1} with the hermitian sphere Σ^{2m+1} of all complex null lines in a hermitian complex vector space W of signature $(m+1, 1)$, with hermitian metric $(\cdot, \cdot)$, by fixing an orthonormal basis of W, i.e. a basis $\{e_0, \cdots, e_{m+1}\}$ with $(e_0, e_0) = -1$, $(e_j, e_j) = 1$, for $j \in \overline{1, m+1}$ and $(e_i, e_j) = 0$ for $i \neq j$, and associating to a complex null line x of W its unique representative of the form $e_0 + u$, where $u \in S^{2m+1}$ belongs to the unit sphere of the positive definite hermitian vector space $\mathrm{Span}\{e_1, \cdots, e_{m+1}\}$. Let η be the natural (line bundle valued) contact form of Σ^{2m+1}:

$$\eta(X) := \mathrm{Im}(\hat{X}w, w), \quad X \in T_x\Sigma^{2m+1}, \quad 0 \neq w \in x, \quad x \in \Sigma^{2m+1},$$

where $\hat{X} \in \mathrm{Hom}_{\mathbb{C}}(x, W)$ is a representative of $X \in \mathrm{Hom}_{\mathbb{C}}(x, W/x)$. A hermitian trace-free endomorphism A of W determines a Reeb vector field T^A of a Sasaki structure on the open subset

$$\Sigma_A^{2m+1} := \{x \in \Sigma^{2m+1} : \quad (Aw, w) > 0, \quad w \in x, \quad w \neq 0\},$$

defined in the following way: at a point $x \in \Sigma_A^{2m+1}$, $T_x^A \in \mathrm{Hom}_{\mathbb{C}}(x, W/x)$ associates to a non-zero vector $w \in x$, the class of iAw in W/x. The contact form of T^A is $\eta_A := \frac{\eta}{(Aw,w)}$, i.e.

$$\eta_A(X) = \frac{\mathrm{Im}(\hat{X}w, w)}{(Aw, w)}, \quad X \in T_x\Sigma^{2m+1}, \quad 0 \neq w \in x, \quad x \in \Sigma_A^{2m+1}.$$

Employing the notations of [8], we shall denote by M_A the induced Kähler structure on the quotient Σ_A^{2m+1}/T^A and by g_A its Kähler metric.

We end this section with a simple Lemma on hermitian operators which will play an important role in our treatment. For completeness of the exposition, we include its proof.

Lemma 2. *Let $A : W \to W$ be a hermitian operator on a complex vector space W with a hermitian metric $(\cdot, \cdot)$ of signature $(m+1, 1)$. Suppose that A satisfies*

$(Aw, w) = 0$, *for any null vector* w *which belongs to a non-empty open subset* D *of* W. *Then* $A = \lambda \mathrm{Id}$, *for* $\lambda \in \mathbb{R}$. *If, moreover,* A *is trace-free, then* $A = 0$.

Proof. Let $w = w_t$ be a curve in D, with w_t null for any t, $w_0 = w \in D$ and $\dot{w}_0 = X$. Taking the derivative at $t = 0$ of the equality $(Aw_t, w_t) = 0$ and using the fact that A is hermitian, we get $\mathrm{Re}(Aw, X) = 0$. In particular, we deduce that $(Aw, X) = 0$, for any null vector $w \in D$ and any $X \in W$, which is hermitian orthogonal to w. This implies that $Aw = \lambda w$, where $\lambda \in \mathbb{R}$ depends a priori on w. It follows that the map

$$W \ni u \to Au \wedge u \in \Lambda^2(W) \tag{4}$$

vanishes when $u \in D$ is null. Being holomorphic, the map (4) must be identically zero. We deduce that for any $u \in W$, Au is a multiple of u which clearly implies the first claim. The second claim is trivial. □

2.4. **The Bryant minimal and characteristic polynomials.** The local type of a Bochner-flat Kähler manifold (M, g, J) is encoded into the Bryant's minimal and characteristic polynomials, which can be defined as follows. Let S be the modified Ricci operator which satisfies $c^*_{\mathcal{K}}(S) = R^g$ (where R^g is the curvature of g) and $P(t)$ be the characteristic polynomial of a new modified Ricci operator Θ, related to S by

$$\Theta := \frac{1}{4}\left(S - \frac{\mathrm{trace}_{\mathbb{R}}(S)}{2(m+2)}\mathrm{Id}\right), \tag{5}$$

where m is the complex dimension of M. The Ricci operator Θ has been introduced by Robert Bryant in [3]. It will be considered as a complex linear operator on the complex vector bundle (TM, J). Its trace is called the modified scalar curvature of (M, g, J). Denote by $\xi_1, \cdots, \xi_l$ the non-constant roots of P and by P_n its non-constant part, defined by $P_n(t) := (t - \xi_1) \cdots (t - \xi_l)$. The number l is called the order of (M, g, J). On a dense open subset M^0 of M, the eigenvalues ξ_j (for any $j \in \{1, \cdots, l\}$) are simple, different from each other at any point and different, at any point, from any constant eigenvalue of Θ; the functions $\xi_1, \cdots, \xi_l$ are functionally independent on M^0 and

$$|\mathrm{grad}_g(\xi_j)|^2 = -4\frac{p_m(\xi_j)}{P'_n(\xi_j)}, \quad j \in \{1, \cdots, l\} \tag{6}$$

where p_m is a monic polynomial of degree $l + 2$, with constant coefficients, independent of j, called the Bryant minimal polynomial of (M, g, J). The Bryant characteristic polynomial p_c of (M, g, J) is by definition the product of p_m with the constant part $P_c := P/P_n$ of P.

Suppose now that $(M, g, J) \cong M_A = \Sigma_A^{2m+1}/T^A$, for a hermitian operator A of W. Denote by $\tilde{a}$ the reduced adjoint operator of A, defined by

$$\tilde{a}(t) = t^{l+2} + a_1 t^{l+1} + \cdots + a_{l+2},$$

where

$$a_k := A^k - \sigma_1(q_A)A^{k-1} + \cdots + (-1)^k \sigma_k(q_A),$$

and $\sigma_k(q_A)$ is the k elementary symmetric function of the roots of the minimal polynomial q_A of A. The reduced adjoint operator $\tilde{a}$ satisfies $(tI - A)\tilde{a}(t) = q_A(t)I$, for any $t \in \mathbb{R}$. The eigenspace of Θ corresponding to a non-constant eigenvalue ξ_j

is spanned by the vector field L_j which, viewed as a section of H, is equal, at a point $x \in \Sigma_A^{2m+1}$, to

$$L_j(w) = \tilde{a}(\xi_j)w \mod w.$$

The non-constant part $P_n(t)$ of the modified Ricci operator Θ of (M, g, J), viewed as a polynomial with function coefficients defined on Σ_A^{2m+1}, is equal, at a point $x \in \Sigma_A^{2m+1}$, to

$$p_{A,x}(t) := \frac{(\tilde{a}(t)w, w)}{(Aw, w)}, \quad w \in x, \quad w \neq 0.$$

The constants eigenvalues of Θ can also be described in terms of A: if λ is a constant eigenvalue of Θ, of multiplicity n, then it is a multiple eigenvalue of A, of multiplicity $n+1$, and the eigenspace of Θ, at a point $x \in \Sigma_A^{2m+1}$, corresponding to λ can be identified with the intersection of the hermitian orthogonal $x^\perp \subset W$ with the eigenspace of A corresponding to λ (see [8]). The Bryant minimal and characteristic polynomials p_m and p_c coincide with the minimal polynomial q_A, respectively to the characteristic polynomial Q_A of A [5], [8]. The modified scalar curvature of M_A, viewed as a function on Σ_A^{2m+1}, is equal, at $x \in \Sigma_A^{2m+1}$, to $-\frac{(A^2w,w)}{(Aw,w)}$, where $w \in x$ is non-zero.

The following Lemma will be useful in our study of generalised Kähler cones and is an easy consequence of the theory developed in [8]. For completeness of the exposition, we include its proof.

Lemma 3. *For any $t \in \mathbb{R}$, $x \in \Sigma_A^{2m+1}$ and $w \in x$ non-zero,*

$$g_A(dp_{A,x}(t), dp_{A,x}(t)) = 4\left(q_A'(t)p_{A,x}(t) - q_A(t)p_{A,x}'(t) - 2tp_{A,x}^2(t) + p_{A,x}^2(t)\frac{(A^2w,w)}{(Aw,w)}\right).$$

Proof. Via the metric g_A, the 1-form $dp_A(t)$ corresponds to the vector field L_t, which, viewed as a section of H, is equal, at $x \in \Sigma_A^{2m+1}$, to the endomorphism

$$L_t(w) := 2\left(\tilde{a}(t)w - p_{A,x}(t)Aw\right) \mod w.$$

Its square norm is equal to

$$\begin{aligned} g_A(L_t, L_t) &= 4\frac{(\tilde{a}(t)w, \tilde{a}(t)w) - 2p_{A,x}(t)(\tilde{a}(t)w, Aw) + p_{A,x}^2(t)(A^2w, w)}{(Aw, w)} \\ &= 4\left(q_A'(t)p_{A,x}(t) - q_A(t)p_{A,x}'(t) - 2tp_{A,x}^2(t) + \frac{p_{A,x}^2(t)(A^2w,w)}{(Aw,w)}\right), \end{aligned}$$

where we have used $(A\tilde{a}(t)w, w) = t(\tilde{a}(t)w, w)$ (which follows from $(tI - A)\tilde{a}(t) = q_A(t)I$ and $(w, w) = 0$) and

$$\frac{(\tilde{a}(t)w, \tilde{a}(t)w)}{(Aw, w)} = q_A'(t)p_{A,x}(t) - q_A(t)p_{A,x}'(t), \tag{7}$$

which has been proved in [8].

□

3. Definition of generalised Kähler cones; the Bochner-flatness condition

Let (N, H, I) be an oriented CR manifold and $\{T_r\}_{r\in\mathcal{J}}$, with $\mathcal{J} \subseteq \mathbb{R}^{>0}$ a connected open interval, a family of Reeb vector fields of Sasaki structures on (N, H, I), with contact forms $\{\theta\}_{r\in\mathcal{J}}$. Let $\omega_r := \frac{1}{2}d\theta_r \in \Lambda^2(N)$ and $g_r := \omega_r(\cdot, I\cdot)$ be the corresponding (positive definite) metrics of the contact bundle H. On the cone manifold $N \times \mathcal{J}$ define the vector fields T, V, a complex structure J and a 2-form ω as in the Introduction, and also a symmetric bilinear form $g := \omega(\cdot, J\cdot)$.

Let $M \subset N \times \mathcal{J}$ be the open subset where $g(T, T)$ is positive. Define a positive function $f : M \to \mathbb{R}^{>0}$ by $g(T, T) = r^2 f$. Note that the restriction $f_r := f(\cdot, r)$ of f to $N_r := M \cap N \times \{r\}$ is positive. We introduce a new family of contact forms $\tilde{\theta}_r = \frac{1}{f_r}\theta_r$; for any r, the contact form $\tilde{\theta}_r$ is defined on N_r (viewed as an open subset of N).

Conventions of notations: For a function $h : M \to \mathbb{R}$, we shall denote by $\dot{h} : M \to \mathbb{R}$ its derivative with respect to r and by $h_r : N_r \to \mathbb{R}$ the restriction of h to a level set N_r.

Lemma 4. *The following equalities hold:*

$$f = 1 + \frac{r\dot{\theta}_r(T_r)}{2} \tag{8}$$

and

$$\dot{\tilde{\theta}}_r = -\frac{2G}{r}\tilde{\theta}_r, \tag{9}$$

where

$$G : M \to \mathbb{R}, \quad G := \frac{r\dot{f}}{2f} - f + 1.$$

Proof. Note that $\omega = \frac{1}{2}(dr \wedge d^J r + r dd^J r)$. It is straightforward to see that $d^J r = r\theta$, where $\theta \in \Lambda^1(N \times \mathcal{J})$ is defined by $\theta(Z) := \theta_r(\pi_* Z)$, for a tangent vector $Z \in T_{(p,r)}(N \times \mathcal{J})$, where $\pi : N \times \mathcal{J} \to N$ is the natural projection. At a tangent space $T_{(p,r)}(N \times \mathcal{J}) = T_p N \times \mathbb{R}$,

$$d\theta = d\theta_r + dr \wedge \dot{\theta}_r$$

and then, restricted to the same tangent space,

$$\omega = r\left(dr \wedge \left(\theta_r + \frac{r}{2}\dot{\theta}_r\right) + \frac{r}{2}d\theta_r\right). \tag{10}$$

It follows that

$$r^2 f = \omega(V, T) = r^2\left(1 + \frac{r\dot{\theta}_r(T_r)}{2}\right),$$

which implies (8). To prove (9), we take the derivative with respect to r of the equality $\tilde{\theta}_r = \frac{1}{f_r}\theta_r$ and we use the fact that $\dot{\theta}_r = \dot{\theta}_r(T_r)\theta_r$. We get

$$\dot{\tilde{\theta}}_r = -\frac{\dot{f}_r}{f_r}\tilde{\theta}_r + \frac{1}{f_r}\dot{\theta}_r = \left(-\frac{\dot{f}_r}{f_r} + \dot{\theta}_r(T_r)\right)\tilde{\theta}_r = -\frac{2G}{r}\tilde{\theta}_r,$$

which proves our Lemma. □

Lemma 5. *The pair (ω, J) defines a Kähler structure on M.*

Proof. From relation (3), it is clear that the integrability tensor N^J of the almost complex structure J, applied to a pair of sections (X, Y) of H, vanishes. On the other hand, $N^J(X, V)$, restricted to a level set $N \times \{r\}$, is equal to $-L_{T_r}(I)(X)$, which is zero, because T_r is the Reeb vector field of a Sasaki structure. It follows that J is integrable. From (10) it is easy to see that T is hermitian orthogonal to H and that the restriction of g to $H \subset T_{(p,r)}(N \times \mathcal{J})$, coincides with $r^2 g_r$, which is positive definite. We deduce that g is positive definite on the subset M of $N \times \mathcal{J}$, where $g(T, T) > 0$, and that (M, ω, J) is a Kähler manifold (the 2-form ω being closed). □

Definition 6. *The Kähler manifold (M, ω, J) is a generalised Kähler cone over the CR manifold (N, H, I). It is a restricted generalised Kähler cone if the function f is constant along the trajectories of the vector field T.*

Convention: For simplicity, in this paper we will consider only restricted generalised Kähler cones; when we refer to a generalised Kähler cone, we will actually mean restricted generalised Kähler cone; this is true also for the statement of Theorem 1.

Remark 1. Main class of generalised Kähler cones: We shall be mainly concerned with generalised Kähler cones over (open subsets) of hermitian CR spheres. Suppose that $N \subset \Sigma^{2m+1}$ is an open subset of the hermitian CR sphere of complex null lines in $W = \mathbb{C}^{m+1,1}$. Then $\theta_r = \frac{\eta}{(B_r w, w)}$, $\tilde{\theta}_r = \frac{\eta}{(A_r w, w)}$ for some hermitian trace-free operators A_r, B_r of W. The condition $T(f) = 0$ is equivalent to $[A_r, B_r] = 0$ for any r, as the following Lemma shows:

Lemma 7. (1) *The operators A_r and B_r are related in the following way:*

$$A_r = B_r - \frac{r}{2}\dot{B}_r. \tag{11}$$

(2) *The functions f and G have the following expression: for any $(x, r) \in M$,*

$$f_r(x) = \frac{(A_r w, w)}{(B_r w, w)}, \quad G(x, r) = \frac{r(\dot{A}_r w, w)}{2(A_r w, w)}; \quad w \in x, \quad w \neq 0.$$

(3) *The condition $T(f) = 0$ is equivalent to $[A_r, B_r] = 0$ for any r.*

Proof. Note that $\dot{\theta}_r = -\frac{(\dot{B}_r w, w)}{(B_r w, w)}\theta_r$ and $\dot{\theta}_r(T_r) = -\frac{(\dot{B}_r w, w)}{(B_r w, w)}$. It follows that

$$f = 1 + \frac{r\dot{\theta}_r(T_r)}{2} = 1 - \frac{r}{2}\frac{(\dot{B}_r w, w)}{(B_r w, w)} = \frac{\left(B_r w - \frac{r}{2}\dot{B}_r w, w\right)}{(B_r w, w)}. \tag{12}$$

We deduce that

$$\tilde{\theta}_r = \frac{1}{f_r}\theta_r = \frac{(B_r w, w)}{\left(B_r w - \frac{r}{2}\dot{B}_r w, w\right)} \cdot \frac{\eta}{(B_r w, w)} = \frac{\eta}{(B_r w - \frac{r}{2}\dot{B}_r w, w)}. \tag{13}$$

Relation (11) follows from (13), $\tilde{\theta}_r = \frac{\eta}{(A_r w, w)}$ and Lemma 2. From (11) and (12) we get the expression of f. On the other hand, from Lemma 4 and $\tilde{\theta}_r = \frac{\eta}{(A_r w, w)}$, we have

$$\dot{\tilde{\theta}}_r = -\frac{2G}{r}\tilde{\theta}_r = -\frac{(\dot{A}_r w, w)}{(A_r w, w)}\tilde{\theta}_r,$$

which implies that G is of the required form. To prove the last statement, note that

$$
\begin{aligned}
T_r(f_r) &= \frac{[(A_rT_rw,w)+(A_rw,T_rw)](B_rw,w)-[(B_rT_rw,w)+(B_rw,T_rw)](A_rw,w)}{(B_rw,w)^2}\\
&= \frac{i([A_r,B_r]w,w)}{(B_rw,w)},
\end{aligned}
$$

since, at a point $x \in N$, $T_r(x) \in \mathrm{Hom}(x, x^{\perp}/x)$ is the homomorphism $T_rw = iB_rw$ mod$w$ and the operators A_r and B_r are hermitian. We conclude from Lemma 2. □

Lemma 8. *The Levi-Civita connection D^g of a generalised Kähler cone (M, ω, J) has the following expression:*

$$
\begin{aligned}
D^g_XY &= \nabla^r_XY - \omega_r(X,Y)T_r - g_r(X,Y)V\\
D^g_VY &= fY + \frac{Y(f)}{2f}V - \frac{(JY)(f)}{2f}T\\
D^g_TY &= \mathcal{L}_T(Y) + fJY + \frac{Y(f)}{2f}T + \frac{(JY)(f)}{2f}V\\
D^g_XV &= fX + \frac{X(f)}{2f}V - \frac{(JX)(f)}{2f}T\\
D^g_VV &= -\frac{1}{2}v + (G+f)V\\
D^g_TV &= \frac{1}{2}Jv + (G+3f-2)T.
\end{aligned}
$$

Here $X, Y \in \Gamma(H)$, the vector field D^g_XY is restricted to a level set N_r, ∇^r is the Tanaka connection of the contact form θ_r of the CR manifold (N, H, I) and v is a vector field on M which belongs, at any point $(p,r) \in M$, to $H_p \subset T_{(p,r)}M$ and is determined by the condition:

$$
g_r(X, v_{(p,r)}) = df(X), \quad \forall X \in H_p \subset T_{(p,r)}M. \tag{14}
$$

Proof. The proof is a straightforward computation based on the Koszul formula. It uses the expression of the Tanaka connection on the contact bundle H, mentioned in Section 2. □

Lemma 9. *The curvature R^g of a generalised Kähler cone (M,ω,J) has the following expression:*

$$\begin{aligned} g(R^g_{X,T}Y,Z) &= -\frac{Y(f)}{2}\omega(X,Z) - \frac{(JY)(f)}{2}g(X,Z) + \frac{Z(f)}{2}\omega(X,Y) \\ &\quad + \frac{(JZ)(f)}{2}g(X,Y) - X(f)\omega(Y,Z) \\ g(R^g_{X_1,X_2}Y,Z) &= g(R^{\nabla^r}_{X_1,X_2}Y,Z) - \frac{f}{r^2}\{g(X_1,Y)g(X_2,Z) - g(X_2,Y)g(X_1,Z)\} \\ &\quad + \frac{f}{r^2}\{-\omega(X_1,Y)\omega(X_2,Z) + \omega(X_2,Y)\omega(X_1,Z) - 2\omega(X_1,X_2)\omega(Y,Z)\} \\ g(R^g_{X,T}V,Y) &= \frac{r^2}{2}(\nabla^r df)^{J,-}(X,JY) + \frac{r^2 f}{2}\left(\nabla^r\left(\frac{df}{f}\right)\right)^{J,+}(X,JY) \\ &\quad + f(2-G-2f)\omega(X,Y) \\ g(R^g_{T,V}V,T) &= g(v,v) + r^2 f\left((G-2)(8f-2) + r\dot{G} + 12f^2\right) \\ g(R^g_{T,V}V,Z) &= -\frac{r^3}{2}d\dot{f}(JZ) + r^2(G-1)df(JZ), \end{aligned}$$

where $X,X_1,X_2,Y,Z\in\Gamma(H)$, $g(R^g_{X_1,X_2}Y,Z)$ is restricted to a level set N_r, v is the vector field defined by (14) and the superscripts $J,+$ and $J,-$ denote the J-invariant part, respectively the J-anti-invariant part of a bilinear form.

Proof. The proof is a lenghty but straightforward computation. □

Proposition 10. *The generalised Kähler cone (M,ω,J) is Bochner-flat if and only if the CR manifold (N,H,I) is flat and the following two conditions hold:*

1. *The function G depends only on r.*
2. *For every $r\in\mathcal{J}$, the contact form $\tilde{\theta}_r = \frac{1}{f_r}\theta_r$ is the contact form of a Sasaki structure on (N_r,H,I), which determines an Einstein Kähler structure on the quotient $N_r/\tilde{T}_r$ (where $\tilde{T}_r$ is the Reeb vector field of $\tilde{\theta}_r$), with modified Ricci tensor*
$$\tilde{S}_r = \left(\frac{r\dot{G}}{2} - G + 2\right)\mathrm{Id}.$$

Proof. The Kähler manifold (M,ω,J) is Bochner flat if and only if

$$R^g = c^*_{\mathcal{K}}(S), \tag{15}$$

for a tensor field $S\in\mathrm{Sym}^{1,1}(M)$. Plugging into (15) the arguments (T,V,V,T) and (T,V,V,Z), for $Z\in\Gamma(H)$, and using formula (2) for the adjoint of the Ricci contraction, we readily deduce that $S(T,T)$ and $S(Z,T)$ are related to the curvature R^g as follows:

$$S(T,T) = -\frac{1}{2r^2 f}g(R^g_{T,V}V,T);\quad S(Z,T) = -\frac{1}{r^2 f}g(R^g_{T,V}V,Z). \tag{16}$$

On the other hand, from Lemma 9 we know that

$$\begin{aligned} g(R^g_{X,V}Y,Z) &= \frac{Y(f)g(X,Z)}{2} - \frac{Z(f)g(X,Y)}{2} - \frac{(JY)(f)\omega(X,Z)}{2} \\ &\quad + \frac{(JZ)(f)\omega(X,Y)}{2} - (JX)(f)\omega(Y,Z), \end{aligned}$$

for every $X,Y,Z\in\Gamma(H)$, and, since $c^*_{\mathcal{K}}(S)(X,V,Y,Z)=g(R^g_{X,V}Y,Z)$, we readily get $S(Z,T)=2(JZ)(f)$. Combining this with the second relation (16), we deduce that

$$-\frac{1}{r^2f}g(R^g_{T,V}V,Z)=2(JZ)(f),$$

which is equivalent, using the expression of $g(R^g_{T,V}V,Z)$ provided by Lemma 9, to $(JZ)(G)=0$. Since $Z\in\Gamma(H)$ is arbitrary, we obtain the first condition of the Proposition (since $X(G)=Y(G)=0$, for $X,Y\in\Gamma(H)$, also $[X,Y](G)=0$; recall now that vector fields of the form $\{X,[X,Y],X,Y\in\Gamma(H)\}$ span the entire TN). To obtain the second condition of the Proposition, we notice that the expression of $g(R^g_{X_1,X_2}Y,Z)$ found in Lemma 9, combined with (15), imply that the CR manifold (N,H,I) is flat and that on the bundle H restricted to a level set N_r,

$$S^{\perp}=\frac{1}{r^2}\left(S_r-2f\mathrm{Id}\right) \tag{17}$$

where $S^{\perp}:H\to H$ is induced by S followed by g-orthogonal projection and $S_r\in\mathrm{End}(H)$ is the modified Ricci tensor of the Kähler curvature $R^{\nabla^r}\in\Lambda^{1,1}(H)\odot\Lambda^{1,1}(H)$ of ∇^r. Plugging into (15) the argument (X,T,V,Y) and using relation (17), we obtain

$$\begin{aligned}g(R^g_{X,T}V,Y)&=\frac14\left(S(JX,Y)g(V,V)+\omega(X,Y)S(V,V)\right)\\&=\frac f4 g(S_r(JX),Y)-\frac14\left(2f^2+\frac{g(R^g_{T,V}V,T)}{2r^2f}\right)\omega(X,Y).\end{aligned}$$

Using the expression of $g(R^g_{X,T}V,Y)$ provided by Lemma 9 we deduce that

$$\begin{aligned}g_r(S_rX,Y)=&-\frac2f\nabla^r(df)^{J,-}(X,Y)-2\nabla^r\left(\frac{df}{f}\right)^{J,+}(X,Y)\\&+\left(4(2-G)-6f+\frac{g(R^g_{T,V}V,T)}{2f^2r^2}\right)g_r(X,Y).\end{aligned}$$

This relation clearly implies that $\nabla^r(df)^{J,-}|_{H\times H}=0$, which means that $\tilde\theta_r=\frac{1}{f_r}\theta_r$ (for any r) is the contact form of a Sasaki structure [11]. Moreover, the modified Ricci tensor $\tilde S_r$ of the the Sasaki structure determined by $\tilde\theta_r$ is related to S_r in the following way [7]

$$\frac1f g_r(\tilde S_r(X),Y)=g_r(S_r(X),Y)+2\nabla^r\left(\frac{df}{f}\right)^{J,+}(X,Y)-\frac{g_r(v,v)}{2f^2}g_r(X,Y). \tag{18}$$

We deduce, using the previous expression of $g_r(S_r(X),Y)$, that

$$\frac1f g_r(\tilde S_r(X),Y)+\left(\frac{g_r(v,v)}{2f^2}+4(G-2)+6f-\frac{g(R^g_{T,V}V,T)}{2r^2f^2}\right)g_r(X,Y)=0.$$

Using again Lemma 9 for the expression of $g(R^g_{T,V}V,T)$ we obtain the second condition of the Proposition. Conversely, it is easy to check that the two conditions of the Proposition ensure the Bochner-flatness of (M,ω,J).

□

The main result of this Section is the following:

Proposition 11. *Let (M, g, J) be a Bochner-flat generalised Kähler cone of complex dimension $m+1 \geq 3$, defined by a family of Sasaki Reeb vector fields $\{T_r\}$ over a CR manifold (N, H, I). Then (N, H, I) is locally isomorphic to the standard CR sphere Σ^{2m+1} of complex null lines in a complex hermitian vector space W of signature $(m+1, 1)$, and $\{T_r\}$ is defined by one of the following families of hermitian operators B_r of W:*

1. *$B_r = r^2(B - \mu(r^2)A)$. Here the real function μ satisfies $\mu' > 0$ and is a solution of the differential equation*

$$\mu' = \frac{1}{2}\mu^2 + d, \tag{19}$$

where $d \in \mathbb{R}$ is an arbitrary real number. The operator A is a hermitian semi-simple operator, with a positive definite eigenspace, of dimension $m+1$, which corresponds to the eigenvalue $\frac{1}{2(m+2)}$ and a 1-dimensional timelike eigenspace, which corresponds to the eigenvalue $-\frac{(m+1)}{2(m+2)}$.

2. *$B_r = r^2(B + \mu(r^2)A)$, where μ satisfies (19) and $\mu' < 0$. The operator A is semi-simple, with an eigenspace of signature $(m, 1)$, which corresponds to the eigenvalue $-\frac{1}{2(m+2)}$, and a 1-dimensional spacelike eigenspace, which corresponds to the eigenvalue $\frac{m+1}{2(m+2)}$.*
3. *$B_r = r^2(B - r^2 A)$, where A is 1-step parabolic, with all eigenvalues equal to zero.*
4. *$B_r = r^2\left(B - \frac{e^{\lambda r^2}}{\lambda}A\right)$, where $\lambda \in \mathbb{R} \setminus \{0\}$, A is 1-step parabolic with all eigenvalues equal to zero.*

In all these cases, B is any hermitian, trace-free operator of W which commutes with A (see Remark 2).

Proof. Since (N, H, I) is flat (see Proposition 10) and of dimension bigger than three, we can assume, restricting N if necessary, that (N, H, I) is an open subset of the hermitian CR sphere Σ^{2m+1} of complex null lines in a complex hermitian vector space W of signature $(m+1, 1)$ [4]. As explained in Lemma 7, the two families of contact forms $\{\theta_r\}_{r\in\mathcal{J}}$ and $\{\tilde{\theta}_r\}_{r\in\mathcal{J}}$ are generated by two families of hermitian trace-free endomorphism $\{B_r\}_{r\in\mathcal{J}}$ and $\{A_r\}_{r\in\mathcal{J}}$ of W respectively such that, for any r, the operators A_r and B_r commute (see Lemma 7). From Lemma 4 we know that $\dot{\tilde{\theta}}_r = -\frac{2G_r}{r}\tilde{\theta}_r$. Since G depends only on r, we get $\tilde{\theta}_r = e^{-\int_{r_0}^{r}\frac{2G_q}{q}dq}\tilde{\theta}_{r_0}$ and we infer that the modified Ricci tensor $\tilde{S}_r$ of $\tilde{\theta}_r$ has the expression $\tilde{S}_r = e^{\int_{r_0}^{r}\frac{2G_q}{q}dq}\tilde{S}_{r_0}$. The second condition of Theorem 10 is equivalent with

$$(\frac{r\dot{G}}{2} - G + 2)\mathrm{Id} = e^{\int_{r_0}^{r}\frac{2G_s}{s}ds}\tilde{S}_{r_0}$$

and implies

$$\left(\frac{r\dot{G}}{2} - G + 2\right)' = \frac{2G}{r}\left(\frac{r\dot{G}}{2} - G + 2\right). \tag{20}$$

Equation (20) can be solved as follows: define a real function μ in the following way:

$$\mu(t) = \frac{G(\sqrt{t}) - 2}{t}. \tag{21}$$

We shall write equation (20) in terms of μ. For this, we first take the derivative of $r^2\mu(r^2) = G(r) - 2$ and we get:

$$\dot{G}(r) = 2r\mu(r^2) + 2r^3\mu'(r^2).$$

It easily follows that

$$\frac{r\dot{G}}{2} - G + 2 = r^4\mu'(r^2). \tag{22}$$

Equation (20) becomes $\mu'' = \mu'\mu$. Since $\mathcal{J}$ is connected, μ satisfies (19), for a constant $d \in \mathbb{R}$. We have the following three possibilities:

(1) $\mu' > 0$. From Lemma 7 we deduce that

$$(\dot{A}_r w, w) = \frac{2}{r}G(r)(A_r w, w) = \frac{2}{r}\left(r^2\mu(r^2) + 2\right)(A_r w, w).$$

Since $\int r\mu(r^2)dr = \frac{1}{2}\ln\left(\mu'(r^2)\right)$ when $\mu' > 0$ we get

$$(A_r w, w) = K(w)r^4\mu'(r^2),$$

where $K = K(w)$ depends only on w. Equivalently,

$$A_r = \mu'(r^2)r^4 A, \tag{23}$$

where $A \in \mathrm{End}(W)$ is hermitian, trace-free, satisfies $(Aw, w) > 0$, for $w \in x$ non-zero, when $(x, r) \in M$. Moreover, (22) together with the second condition of Proposition 10 imply that the modified Ricci tensor S_A of the Kähler-Einstein structure M_A is the identity endomorphism, from where we deduce that A is as in the statement of the Proposition (see [8]). On the other hand, from Lemma 7, B_r must satisfy (11), with $A_r = \mu'(r^2)r^4A$. It follows that

$$B_r = r^2\left(B - \mu(r^2)A\right),$$

where $B \in \mathrm{End}(W)$ is hermitian and trace free.

(2) $\mu' < 0$. Then $\int r\mu(r^2)dr = \frac{1}{2}\ln\left(-\mu'(r^2)\right)$. A similar argument shows that

$$A_r = -r^4\mu'(r^2)A$$
$$B_r = r^2\left(B + \mu(r^2)A\right),$$

but in this case the Bochner-flat Kähler structure M_A has the modified Ricci operator $S_A = -\mathrm{Id}$, which implies that A is as in the statement of the Proposition (see [8]).

(3) It remains to consider the case when the function μ is constant. Then $\mu(t) = \lambda$ for $\lambda \in \mathbb{R}$, $G(x, r) = \lambda r^2 + 2$ and

$$(\dot{A}_r w, w) = \frac{2G}{r}(A_r w, w) = \frac{2(\lambda r^2 + 2)}{r}(A_r w, w).$$

We distinguished two subcases: (i) $\lambda = 0$; (ii) $\lambda \neq 0$. In subcase (i) we obtain

$$A_r = r^4A, \quad B_r = r^2(B - r^2A),$$

and in subcase (ii),

$$A_r = r^4e^{\lambda r^2}A, \quad B_r = r^2\left(B - \frac{e^{\lambda r^2}}{\lambda}A\right).$$

Since $\frac{r\dot{G}}{2} - G + 2 = 0$, the Kähler structure M_A is flat and hence the endomorphism A is 1-step parabolic, with all eigenvalues zero (see [8]).

In all cases (1), (2) and (3), the generalised Kähler cone condition $T(f) = 0$ becomes $[A, B] = 0$ (see Lemma 7).

□

Remark 2. The condition $[A, B] = 0$ of Proposition 11 determines the operator B as follows:

(1) In the first case of Proposition 11, B preserves, up to a multiplicative constant, a timelike eigenvector v (unique, up to a non-zero multiplicative constant) of A. On the hermitian orthogonal $v^{\perp}$, the hermitian metric $(\cdot, \cdot)$ is positive definite, $A : v^{\perp} \to v^{\perp}$ is a multiple of the identity endomorphism and $B : v^{\perp} \to v^{\perp}$, being hermitian, is diagonalisable. It follows that A and B are simultaneously diagonalisable.

(2) In the second case of Proposition 11, B preserves, up to a multiplicative constant, a spacelike eigenvector v (unique, up to a non-zero multiplicative constant) of A, which corresponds to the eigenvalue $\frac{m+1}{2(m+2)}$. On the hermitian orthogonal $v^{\perp}$, the hermitian metric $(\cdot, \cdot)$ has signature $(m, 1)$, A is a multiple of the identity endomorphism and $B : v^{\perp} \to v^{\perp}$ can be elliptic, hyperbolic, 1- or 2-step parabolic.

(3) Consider now the cases three and four of Proposition 11. Note that $A = 0$ on any positive definite eigenspace of B (since $[A, B] = 0$, A preserves such an eigenspace, say W_1, of B; because the hermitian metric $(\cdot, \cdot)$ is positive definite on W_1 and A is hermitian, A is diagonalisable on W_1; this forces it to be zero, because A does not have non-zero eigenvalues). Let us denote by $W_1, \cdots, W_s$, the positive definite eigenspaces of B and by W_0 the hermitian orthogonal of the direct sum $\oplus_{j=1}^{s} W_j$. The eigenspaces W_j (for $j \in \{1, \cdots, s\}$) correspond to eigenvalues, say β_j, of B, which can be any real numbers. It remains to study the restriction B_0 of B to W_0. We notice first that B_0 cannot be hyperbolic: if it was hyperbolic, then B would have two complex non-real eigenvalues, say δ and $\bar{\delta}$, with 1-dimensional eigenspaces, generated by two null independent vectors v_1 and v_2 respectively. However, since $[A, B] = 0$, $BAv_1 = \delta Av_1$ and $BAv_2 = \bar{\delta} Av_2$, which imply that $Av_1 = Av_2 = 0$ (because A has no non-zero eigenvalues). But if we take an orthonormal basis $\{e_0, \cdots, e_n\}$ of W_0 in which

$$A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(which is possible since $A : W_0 \to W_0$ is 1-step parabolic with all eigenvalues equal to zero) the conditions v_1, v_2 null and $Av_1 = Av_2 = 0$ would imply that v_1 and v_2 are multiples of $e_0 + e_1$. In particular they would be dependent, which is a contradiction. We conclude that B_0 can be elliptic or 1- or 2-step parabolic. Therefore, $B_0 = \gamma I + N$, for an endomorphism N of W_0 which commutes with A and which satisfies $N^3 = 0$, and for $\gamma \in \mathbb{R}$ which is different from all β_j. The endomorphism $N^{\perp}$ of $\widehat{W_0} := \mathrm{Span}\{e_2, \cdots, e_n\}$

obtained from N by restriction and orthogonal projection, is hermitian on $\widehat{W_0}$. Because the metric $(\cdot,\cdot)$ is positive definite on $\widehat{W_0}$, $N^{\perp}$ is diagonalisable and hence there is a basis $\{e_2',\cdots,e_n'\}$ of $\widehat{W_0}$, such that $N^{\perp}$ is diagonal in this basis. If we consider now the basis $\mathcal{B} := \{e_0, e_1, e_2', \cdots, e_n'\}$ of W_0, it is straightforward to see that $[A,N]=0$ and N-hermitian imply that

$$N = \begin{pmatrix} \gamma_0 & \alpha & \mu_2 & \mu_3 & \cdots & \mu_n \\ -\alpha & \gamma_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ -\bar\mu_2 & \bar\mu_2 & \gamma_2 & 0 & \cdots & 0 \\ -\bar\mu_3 & \bar\mu_3 & 0 & \gamma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\bar\mu_n & \bar\mu_n & 0 & 0 & \cdots & \gamma_n. \end{pmatrix}$$

in the basis $\mathcal{B}$. Moreover, $N^3=0$ if and only if $\gamma_0=-\gamma_1=-\alpha$ and $\gamma_j=0$ for any $j\in\{2,\cdots,n\}$ and $N^2=0$ if and only $\gamma_0=-\gamma_1=-\alpha$, $\gamma_j=\mu_j=0$, for any $j\in\{2,\cdots,n\}$. Since B is trace-free, the real constants β_j and γ must satisfy $(n+1)\gamma+\sum_{i=1}^{s} n_i\beta_i=0$, where n_i is the dimension of W_i (and $n+1$ is the dimension of W_0).

4. The local geometry of Bochner-flat generalised Kähler cones

In this section we prove our main Theorem 1. We will do this by analysing the local types of the Bochner-flat generalised Kähler cones determined in Proposition 11. The results we shall obtain in this Section can be summarized by the following table:

	$\mu' > 0$	$\mu' = 0$
$d > 0$	(all) hyperbolic	$---$
$d = 0$	(all) $1-$step parabolic	$\lambda = 0$; (all) $2-$step parabolic
$d < 0$	(all) elliptic	$\lambda = \pm\sqrt{-2d}$, (all) $1,2-$step parabolic, elliptic

In particular, we show that all elliptic, hyperbolic, 1 and 2-step parabolic Bochner-flat Kähler manifolds are locally generalised Kähler cones, which proves our main Theorem 1.

Convention: Without further explanations, we employ the notations of the previous section.

4.1. The case 1 of Theorem 11.

In this Subsection we analyze the first column of the above table.

Proposition 12. *Let (M,g,J) be a Bochner-flat generalised Kähler cone which belongs to the first case of Proposition 11. Then (M,g,J) is:*

1. *of hyperbolic type, if $d>0$.*
2. *of elliptic type, if $d<0$.*
3. *of 1-step parabolic type, if $d=0$.*

Conversely, any Bochner-flat Kähler manifold which is of elliptic, hyperbolic or 1-step parabolic type can be locally realised as a generalised Kähler cone, which belongs to the first case of Proposition 11.

Proof. In the first case of Proposition 11,

$$M = \{(x, r) \in \Sigma^{2m+1} \times \mathbb{R}^{>0} : \quad (Bw, w) > \frac{1}{2}\mu(r^2)(Aw, w), (Aw, w) > 0, \forall w \in x, w \neq 0\}.$$

From Remark 2, there is an orthonormal basis $\mathcal{B} = \{e_0, e_1, \cdots, e_{m+1}\}$ of W such that both operators A and B are diagonal in this basis:

$$B = \mathrm{diag}(-k, k_1, \cdots, k_{m+1})$$
$$A = \frac{1}{2(m+2)}\mathrm{diag}(-m-1, 1, \cdots, 1).$$

Here $k_j \in \mathbb{R}$, for any $j \in \{1, \cdots, m+1\}$, and $k = k_1 + \cdots + k_{m+1}$. We shall identify Σ^{2m+1} with the unit sphere S^{2m+1} in $\mathrm{Span}\{e_1, \cdots, e_{m+1}\}$ and $S^{2m+1} \times \mathbb{R}^{>0}$ with $\mathbb{C}^{m+1} \setminus \{0\}$, by means of the diffeomorphism

$$h : S^{2m+1} \times \mathbb{R}^{>0} \to \mathbb{C}^{m+1} \setminus \{0\}, \quad f(z_1, \cdots, z_{m+1}, r) := (rz_1, \cdots, rz_{m+1}).$$

Read on the image

$$h(M) = \{(z_1, \cdots, z_{m+1}) \in \mathbb{C}^{m+1} \setminus \{0\}, \quad \sum_{j=1}^{m+1} \left(k + k_j - \frac{1}{2}\mu(r^2)\right) |z_j|^2 > 0\},$$

the complex structure J, at a point $z \in h(M)$, satisfies

$$J(V) = r^2 \sum_{j=1}^{m+1} \left(k_j + k - \frac{1}{2}\mu(r^2)\right) \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right)$$
$$J_{|z^\perp} = i$$

and the Kähler form ω is equal to $\frac{1}{4}dd^J r^2$. Here $r^2 = |z_1|^2 + \cdots + |z_{m+1}|^2$ and $z_j = x_j + iy_j$ for any $j \in \{1, \cdots, m+1\}$. For simplicity, we restrict the Kähler structure (ω, J) to the subset

$$D := \{z \in \mathbb{C}^{m+1} \setminus \{0\}, \quad \mu(r^2) < 2(k_j + k), \quad j \in \overline{1, m+1}\} \tag{24}$$

of $h(M)$. We shall consider separately the three cases: $d > 0$, $d < 0$ and $d = 0$.

(1) Suppose that $d = \frac{\beta^2}{2}$, with $\beta > 0$. Then $\mu(t) = \beta \mathrm{tg}\left(\frac{\beta t}{2}\right)$. It can be checked that the map

$$F(z_1, \cdots, z_{m+1}) := (w_1 = f_1(r^2)z_1, \cdots, w_{m+1} = f_{m+1}(r^2)z_{m+1}), \tag{25}$$

where

$$f_j(t) = \left(\frac{\beta}{\sqrt{2}}\right)^{1/2} \frac{e^{\frac{(k_j + k)t}{2}}}{t^{1/2}\mu'(t)^{1/4}}, \tag{26}$$

is an isomorphism between the Kähler manifolds $(D, \frac{1}{4}dd^J r^2, J)$ and $(F(D), \frac{1}{4}dd^{J_0}x, J_0)$. Here J_0 is the standard complex structure of $\mathbb{C}^{m+1}$ and the positive function $x = x(w_1, \cdots, w_{m+1})$ is defined by the implicit equation

$$\sum_{j=1}^{m+1} \frac{|w_j|^2}{f_j^2(x)} = x. \tag{27}$$

Let $y = y(w_1, \cdots, w_{m+1})$ be related to x by the formula

$$x = \frac{4}{\beta}\mathrm{arctg}[(1 + y)^{1/2}]$$

and notice that

$$\frac{e^{(k_j+k)x}}{\dot{\mu}(x)^{1/2}} = \frac{\sqrt{2}|y|}{\beta(2+y)} e^{2l_j \mathrm{arctg}[(1+y)^{1/2}]}$$

where $l_j = \frac{2}{\beta}(k_j + k)$, for $j \in \{1, \cdots, m+1\}$. For simplicity, we restrict to the set, say $D' \subset F(D)$, where $y > 0$. On this set, $(\frac{1}{4} dd^{J_0} x, J_0)$ coincides with the Kähler structure $(\frac{1}{\beta} dd^{J_0} \mathrm{arctg}[(1+y)^{1/2}], J_0)$, where y is defined by the implicit equation

$$\sum_{j=1}^{m+1} \frac{|w_j|^2}{e^{2l_j \mathrm{arctg}[(1+y)^{1/2}]}} = \frac{y}{2+y}. \tag{28}$$

The Kähler structure $(\frac{1}{\beta} dd^{J_0} \mathrm{arctg}[(1+y)^{1/2}], J_0)$ is of hyperbolic type (see Section 2 of [10]), isomorphic with M_C, where C a hyperbolic hermitian operator of $\mathbb{C}^{m+2,1}$ with characteristic and minimal polynomials

$$Q_C = \left(\left(t + \frac{2(m+2)k}{\beta(m+3)} \right)^2 + 1 \right) \prod_{j=1}^{m+1} \left(t - \frac{2}{\beta} \left(k_j + \frac{k}{m+3} \right) \right)$$

$$q_C = \left(\left(t + \frac{2(m+2)k}{\beta(m+3)} \right)^2 + 1 \right) \prod_{i=1}^{s} \left(t - \frac{2}{\beta} \left(k_i + \frac{k}{m+3} \right) \right).$$

Here $i \in \{1, \cdots, s\}$ parametrises the distinct values of $\{k_1, \cdots, k_{m+1}\}$. This proves our first claim.

(2) Next, suppose that $d := -\frac{\beta^2}{2}$, where $\beta > 0$. Then $\mu(t) = \frac{\beta(1+e^{t\beta+p})}{1-e^{t\beta+p}}$, where $p \in \mathbb{R}$. By means of the function (25) with

$$f_j(t) = \frac{\beta^{1/2} 2^{1/4} e^{\frac{1}{2}(k_j+k)(t+\frac{p}{\beta})+p}}{t^{1/2} \mu'(t)^{1/4}}, \quad j \in \{1, \cdots, m+1\}, \tag{29}$$

the Kähler manifolds (D, ω, J) and $(F(D), \omega_0 := \frac{1}{4} dd^{J_0} x, J_0)$ are isomorphic, where $x = x(w_1, \cdots, w_{m+1})$ is defined by the implicit equation (27), with functions f_j defined in (29). Define a new function $y = y(w_1, \cdots, w_{m+1})$ by

$$y = e^{\beta x} - e^{-p}.$$

For simplicity, we restrict to the subset of $F(D)$, say D', where $y > 0$. On D', the function y satisfies the implicit equation

$$\sum_{j=1}^{m+1} \frac{|w_j|^2}{e^p y (e^p y + 1)^{\beta_j}} = 1,$$

with $\beta_j := \frac{1}{\beta}(k_j + k - \frac{\beta}{2})$. In terms of y, $\omega_0 = \frac{1}{4\beta} dd^{J_0} \log(e^p y + 1)$. It follows that (ω_0, J_0) is of elliptic type on D' (see Section 2.2 of [8]) and is isomorphic with the Kähler structure M_C, with C a semi-simple hermitian operator of $\mathbb{C}^{m+2,1}$, with eigenvalues $-\frac{\beta}{2} - \frac{(m+2)k}{m+3}, k_1 + \frac{k}{m+3}, \cdots, k_{m+1} + \frac{k}{m+3}, \frac{\beta}{2} - \frac{(m+2)k}{m+3}$. Our second claim follows.

(3) Finally, suppose that $d = 0$. Then $\mu(t) = -\frac{2}{t+q}$, for $q \in \mathbb{R}$. For simplicity, we assume that $q \geq 0$. The function (25), with

$$f_j(t) = \frac{e^{\frac{(k_j+k)t}{2}} (t+q)^{1/2}}{t^{1/2}}$$

defines an isomorphism between the Kähler manifolds (D,ω,J) and $(F(D),\omega_0 := \frac{1}{4}dd^{J_0}x, J_0)$ where $x = x(w_1,\cdots,w_{m+1})$ is a positive function defined implicitly by the equation

$$\sum_{j=1}^{m+1}\frac{|w_j|^2}{e^{(k_j+k)x}(x+q)} = 1.$$

Note that the function $y := x+q > 0$ satisfies

$$\sum_{j=1}^{m+1}\frac{|w_j|^2}{e^{(k_j+k)(y-q)}} = y. \tag{30}$$

Moreover, $\omega_0 = \frac{1}{4}dd^{J_0}y$. The Kähler structure $(\frac{1}{4}dd^{J_0}y, J_0)$ is of 1-step parabolic type (see Section 3.1 of [8]). It is isomorphic to the Kähler structure M_C, where C is a 1-step parabolic hermitian operator of $\mathbb{C}^{m+2,1}$, with characteristic and minimal polynomials

$$Q_C(t) = \left(t+\frac{(m+2)k}{m+3}\right)^2\prod_{j=1}^{m+1}\left(t-k_j-\frac{k}{m+3}\right)$$
$$q_C(t) = \left(t+\frac{(m+2)k}{m+3}\right)^2\prod_{i=1,k_i\neq -k}^{s}\left(t-k_i-\frac{k}{m+3}\right).$$

As before, $i\in\{1,\cdots,s\}$ parametrises the distinct values of $\{k_1,\cdots,k_{m+1}\}$. Our third claim follows.

The last statement of the Proposition follows by an examination of the minimal and characteristic polynomials of the operators C we found in each of the cases (1), (2) and (3).

□

4.2. **The cases 3 and 4 of Proposition 11.** In this Section we prove the following

Proposition 13. (1) *Let (M,g,J) be a Bochner-flat generalised Kähler cone, which belongs to the third case of Proposition 11. Then (M,g,J) is of 2-step parabolic type.*

(2) *Let (M,g,J) be a Bochner-flat generalised Kähler cone, which belongs to the fourth case of Proposition 11. Then (M,g,J) is of 2-step parabolic type, except when $\mu_j = 0$ for any $j\in\{2,\cdots,n\}$. In this case it is of 1-step parabolic type if $\alpha\neq 0$ and of elliptic type if $\alpha = 0$.*

(3) *Any 2-step parabolic Bochner-flat Kähler structure can be locally realised as a Bochner-flat generalised Kähler cone which belongs to the third case of Proposition 11, and also as a Bochner-flat generalised Kähler cone which belongs to the fourth case of Proposition 11.*

We divide the proof into several Lemmas and Propositions. Let (M,ω,J) be a Bochner-flat generalised Kähler cone, which belongs to the third or to the fourth case of Proposition 11. We preserve the notations of Proposition 11 and of Remark 2.

Proposition 14. *The Bryant modified Ricci operator* Θ *of* (M, g, J) *has the following properties:*

$$\Theta(L_j) = \left(\eta_j - \frac{c}{m+3}\right) L_j - \frac{q_r(\xi_j)}{r^4\,(\eta_j - \gamma)} V, \quad j \in \{1, \cdots, l\}$$

$$\Theta(v_k) = \left(\frac{\lambda_k}{r^2} - \frac{c}{m+3}\right) v_k, \quad k \in \{1, \cdots, m-l\}$$

$$\Theta(V) = \frac{f}{r^4} \sum_{j=1}^{l} \frac{L_j}{p_r'(\xi_j)\,(\eta_j - \gamma)} - \left(\frac{f}{r^2} - \frac{(m+2)c}{m+3}\right) V.$$

Here $\xi_1, \cdots, \xi_l$ *(respectively,* $\lambda_1, \cdots, \lambda_{m-l}$*) are the non-constant (respectively, constant) eigenvalues of the Bryant modified Ricci operator* Θ_r *of* M_{B_r}*,* $L_1, \cdots, L_l$ *are vector fields on* M *which, at a point* $(x, r) \in M$*, belong to* $H_x = \mathrm{Hom}_{\mathbb{C}}(x, W/x)$ *and are the homomorphisms* $L_j(w) = \tilde{b}_r(\xi_j)w \mod w$*,* $v_1, \cdots, v_{m-l}$ *are eigenvectors of* Θ *which correspond to the eigenvalues* $\lambda_1, \cdots, \lambda_{m-l}$*,* $p_r(t) = \prod_{j=1}^{l}(t - \xi_j)$ *is the non-constant part of the characteristic polynomial of* Θ_r*,* $\eta_1 := \frac{\xi_1}{r^2}, \cdots, \eta_l := \frac{\xi_l}{r^2}$*,* q_r *is the minimal polynomial of* B_r *and* $c = \gamma - \lambda$*.*

Proof. Recall that Θ is related to the modified Ricci tensor S from the proof of Proposition 10 by

$$\Theta = \frac{1}{4}\left(S - \frac{\mathrm{trace}_{\mathbb{R}}(S)}{2(m+3)}\mathrm{Id}\right).$$

We need to determine $S(L_j)$, $S(v_k)$, $S(V)$ and $\mathrm{trace}_{\mathbb{R}}(S)$. From the proof of Proposition 10 we know that for any $X \in H$,

$$S(X) = \frac{1}{r^2}\left(S_r X - 2fX\right) + \frac{2(JX)(f)}{r^2 f} T - \frac{2X(f)}{r^2 f} V. \tag{31}$$

It is easy to check the following equalities:

$$df(L_j) = \frac{2f q_r(\xi_j)}{r^2(\eta_j - \gamma)}; \quad df(JL_j) = 0; \quad df(v_k) = 0; \quad df(Jv_k) = 0 \tag{32}$$

which imply, using $\Theta_r(L_j) = \xi_j L_j$, $\Theta_r(v_k) = \lambda_k v_k$ and relation (5) applied to Θ_r, that

$$S(L_j) = \frac{2}{r^2}\left(2\xi_j - \frac{(B_r^2 w, w)}{(B_r w, w)} - f\right) L_j - \frac{4 q_r(\xi_j)}{r^4\,(\eta_j - \gamma)} V$$

$$S(v_k) = \frac{2}{r^2}\left(2\lambda_k - \frac{(B_r^2 w, w)}{(B_r w, w)} - f\right) v_k.$$

To evaluate $S(V)$, we write it as a sum of $S^{\perp}(V)$, the g-orthogonal projection of $S(V)$ on H, and $\frac{S(V,V)}{r^2 f} V$. It is easy to check, using the fact that S is hermitian, relation (6) and $L_j = \frac{p_r'(\xi_j)}{2}\mathrm{grad}_{g_r}(\xi_j)$ (see Theorem 2 of [8]), that

$$S^{\perp}(V) = \frac{4f}{r^4} \sum_{j=1}^{l} \frac{L_j}{p_r'(\xi_j)(\eta_j - \gamma)}. \tag{33}$$

To evaluate $S(V, V) = S(T, T)$, we use the first equality of (16) and the expression of $g(R^g_{T,V} V, T)$ provided by Lemma 9. Notice that the vector field v defined in Lemma

8 has the following expression: at a point $(x,r)\in M$, $v_{(x,r)}\in H_x=\mathrm{Hom}(x,W/x)$ is equal to

$$v_{(x,r)}(w)=2\left(A_r w-f_r(x)B_r w\right)\quad \mathrm{mod} w,$$

and so

$$g_r(v_{(x,r)},v_{(x,r)})=4f_r(x)^2\left(\frac{(B_r^2w,w)}{(B_rw,w)}-2r^2\gamma\right),\tag{34}$$

where $w\in x$ is non-zero. It follows that

$$S(V,V)=-2f\left(\frac{(B_r^2w,w)}{(B_rw,w)}-2r^2\gamma\right)-4\lambda r^2f-6f^2.\tag{35}$$

Relation (33), together with (35), determine $S(V)$. It remains to calculate $\mathrm{trace}_{\mathbb{R}}(S)$. Using (31) and (35), we have

$$\begin{aligned}\mathrm{trace}_{\mathbb{R}}(S)&=\frac{1}{r^2}\left(\mathrm{trace}_{\mathbb{R}}(S_r)-4mf\right)+\frac{2}{r^2f}S(V,V)\\&=-\frac{4}{r^2}\left((m+3)\left(\frac{(B_r^2w,w)}{B_rw,w)}+f\right)-2r^2c\right).\end{aligned}$$

Our claim follows now easily, combining the expressions of $S(L_j)$, $S(v_k)$, $S(V)$ and $\mathrm{trace}_{\mathbb{R}}(S)$ determined above.

□

We introduce a new family of hermitian operators $\hat{B}_r:=\frac{1}{r^2}B_r=B+\delta(r)A$; $\delta(r):=-r^2$ when $\lambda=0$ (equivalently, when (M,ω,J) belongs to the third case of Proposition 11) and $\delta(r)=-\frac{e^{\lambda r^2}}{\lambda}$ when $\lambda\neq 0$ (equivalently, when (M,ω,J) belongs to the fourth case of Proposition 11). Let $\hat{q}$, respectively $\hat{Q}$, be the minimal and characteristic polynomials of $\hat{B}_r$, equal to

$$\hat{q}(t)=(t-\gamma)^3\prod_{j=1}^{s}(t-\beta_j);\quad \hat{Q}:=(t-\gamma)^{n+1}\prod_{j=1}^{s}(t-\beta_j)^{n_j}$$

if there is $\mu_j\neq 0$ and

$$\hat{q}(t)=(t-\gamma)^2\prod_{j=1}^{s}(t-\beta_j);\quad \hat{Q}(t)=(t-\gamma)^{n+1}\prod_{j=1}^{s}(t-\beta_j)^{n_j}$$

otherwise. Let $\hat{p}_r(t):=\prod_{j=1}^{l}(t-\eta_j)$ be the non-constant part of the characteristic polynomial of the Bryant modified Ricci operator $\hat{\Theta}_r$ of the Kähler structure $M_{\hat{B}_r}$. It will be considered as a polynomial with function coefficients defined on $\Sigma^{2m+1}_{\hat{B}_r}$. We shall denote by $\hat{p}_{r,x}$ its value at a null line $x\in\Sigma^{2m+1}_{\hat{B}_r}$, which is a polynomial with constant coefficients.

Proposition 15. *Let $\hat{q}_1$ be the constant polynomial equal to the quotient of $\hat{q}$ by $(t-\gamma)^2$. Then the characteristic polynomial $P(t)$ of the modified Ricci operator Θ of (M,g,J) is equal to the product $\frac{\hat{Q}\left(t+\frac{c}{m+3}\right)}{\hat{q}\left(t+\frac{c}{m+3}\right)}P_1(t)$, where*

$$P_1(t):=\left(t-\frac{(m+2)c}{m+3}\right)\hat{p}_r\left(t+\frac{c}{m+3}\right)+\frac{f}{r^2}\hat{q}_1\left(t+\frac{c}{m+3}\right).\tag{36}$$

Proof. Using Proposition 14, together with the fact that the constant part of the characteristic polynomial of the modified Ricci operator Θ_A of a Bochner-flat Kähler structure M_A is equal to the quotient of the characteristic polynomial by the minimal polynomial of A (see [8], Section 1.5), it is easy to see that $P(t) = \frac{\hat{Q}\left(t+\frac{c}{m+3}\right)}{\hat{q}\left(t+\frac{c}{m+3}\right)} P_1(t)$, where

$$P_1(t) = \left(t - \frac{(m+2)c}{m+3} + \frac{f}{r^2}\right) \hat{p}_r \left(t + \frac{c}{m+3}\right) + \frac{f}{r^8} \sum_{j=1}^{l} \frac{q_r(\xi_j)}{(\eta_j - \gamma)^2 p_r'(\xi_j)} \prod_{i \neq j} \left(t - \eta_i + \frac{c}{m+3}\right).$$

We shall evaluate the expression

$$\mathcal{E}(t) := \frac{f}{r^8} \sum_{j=1}^{l} \frac{q_r(\xi_j)}{(\eta_j - \gamma)^2 p_r'(\xi_j)} \prod_{i \neq j} \left(t - \eta_i + \frac{c}{m+3}\right).$$

For this, let $g_r, \hat{g}_r$ be the Bochner-flat Kähler metrics of M_{B_r}, respectively, $M_{\hat{B}_r}$ (viewed as metrics on H). Then, from relation (10), $\hat{g}_r = r^2 g_r = g$ on H. Using (6), we get

$$-4\frac{q_r(\xi_j)}{p_r'(\xi_j)} = g_r\left(\operatorname{grad}_{g_r}(\xi_j), \operatorname{grad}_{g_r}(\xi_j)\right) = r^6 \hat{g}_r\left(\operatorname{grad}_{\hat{g}_r}(\eta_j), \operatorname{grad}_{\hat{g}_r}(\eta_j)\right) = -4r^6 \frac{\hat{q}(\eta_j)}{\hat{p}_r'(\eta_j)},$$

which implies that

$$\mathcal{E}(t) = \frac{f}{r^2} \sum_{j=1}^{l} \frac{\hat{q}_1(\eta_j)}{\hat{p}_r'(\eta_j)} \prod_{i \neq j} \left(t - \eta_i + \frac{c}{m+3}\right).$$

Note that $\mathcal{E}_1(t) := \frac{r^2}{f}\mathcal{E}(t)$ is a polynomial of degree $l-1$ which satisfies

$$\mathcal{E}_1\left(\eta_j - \frac{c}{m+3}\right) = \frac{\hat{q}_1(\eta_j)}{\hat{p}_r'(\eta_j)} \prod_{i \neq j} (\eta_j - \eta_i) = \hat{q}_1(\eta_j), \quad j \in \{1, \cdots, l\}$$

Since $\hat{q}_1$ is a monic polynomial of degree l and $\mathcal{E}_1$ is of degree $l-1$, we deduce that $\hat{q}_1(t) = \mathcal{E}_1\left(t - \frac{c}{m+3}\right) + \hat{p}_r(t)$ which implies that

$$\sum_{j=1}^{l} \frac{\hat{q}_1(\eta_j)}{\hat{p}_r'(\eta_j)} \prod_{i \neq j} \left(t - \eta_i + \frac{c}{m+3}\right) = \hat{q}_1\left(t + \frac{c}{m+3}\right) - \hat{p}_r\left(t + \frac{c}{m+3}\right). \tag{37}$$

Our claim follows.

□

The possible constant roots of the polynomial P_1 (which are also constant eigenvalues of Θ) are determined in Proposition 17. In the proof of this Proposition we will need the following additional Lemma.

Lemma 16. *The following equality holds:*

$$\frac{d}{dr}\hat{p}_r(t) = \frac{2f}{r}\left(\hat{p}_r(t) - \hat{q}_1(t)\right).$$

Proof. We take the derivative with respect to r of the equality

$$\big((\eta_j I - B - \delta A)^{-1} w, w\big) = 0$$

(which follows from $\hat{p}_r(\eta_j) = 0$) and we obtain

$$\Big((\eta_j I - \hat{B}_r)^{-1}(\dot{\eta}_j I - \dot{\delta} A)(\eta_j I - \hat{B}_r)^{-1} w, w\Big) = 0. \tag{38}$$

On the other hand, since $AB = \gamma A$, it is easy to see that

$$(\eta_j I - \hat{B}_r)^{-1} A (\eta_j I - \hat{B}_r)^{-1} = \frac{A}{(\eta_j - \gamma)^2}. \tag{39}$$

Applying (7) to $A := \hat{B}_r$ and using the fact that $\eta_j I - \hat{B}_r$ is invertible, we get

$$\Big((\eta_j I - \hat{B}_r)^{-2} w, w\Big) = -\frac{\hat{p}_r'(\eta_j)}{\hat{q}(\eta_j)} (\hat{B}_r w, w). \tag{40}$$

Combining (38), (39) and (40), and using the fact that $\delta(r) = -\frac{e^{\lambda r^2}}{\lambda}$ when $\lambda \neq 0$ and $\delta(r) = -r^2$ when $\lambda = 0$, we deduce the expressions of the derivatives $\dot{\eta}_j$ as follows:

$$\dot{\eta}_j = \frac{2f}{r} \frac{\hat{q}_1(\eta_j)}{\hat{p}_r'(\eta_j)}. \tag{41}$$

Since $\frac{d}{dr}\hat{p}_r(t) = -\sum_{j=1}^{l} \dot{\eta}_j \prod_{i\neq j}(t - \eta_i)$ we get, using (37), our claim. $\square$

Proposition 17. *The following statements hold:*

1. *Suppose that $\hat{q}_1(c) \neq 0$. Then the polynomial P_1 defined in (36) does not have constant roots, except when $\lambda \neq 0$ and $\alpha = \mu_j = 0$ for any j. In this case, $t := \frac{(m+2)c}{m+3} + \lambda$ is the unique constant root of P_1 and is simple.*
2. *Suppose that $\hat{q}_1(c) = 0$. Then $t := \frac{(m+2)c}{m+3}$ is a simple root of P_1. The polynomial P_1 has other constant roots if and only if $\lambda \neq 0$ and $\alpha = \mu_j = 0$ for any j. In this case, there is only one additional constant root of P_1, namely $t := \lambda + \frac{(m+2)c}{m+3}$, which is simple.*

Proof. We first consider the case when $\hat{q}_1(c) \neq 0$. We claim that P_1 has no multiple roots. Suppose, on the contrary, that t is a multiple (necessarily constant, because the non-constant eigenvalues of the Bryant modified Ricci operator are always simple) root of P_1. Since $\hat{q}_1(c) \neq 0$, t cannot be equal to $\frac{(m+2)c}{m+3}$ and so $\hat{q}_1\left(t + \frac{c}{m+3}\right) \neq 0$ (because $\hat{p}_r$ has no constant roots). The equalities $P_1(t) = P_1'(t) = 0$ imply that

$$t_1 \hat{p}_r'\left(t + \frac{c}{m+3}\right) + t_2 \hat{p}_r\left(t + \frac{c}{m+3}\right) = 0, \tag{42}$$

where $t_1 := t - \frac{(m+2)c}{m+3} \in \mathbb{R} \setminus \{0\}$ and $t_2 := -\frac{(m+2)c}{m+3} - \left(t - \frac{(m+2)c}{m+3}\right) \frac{\hat{q}_1'\left(t + \frac{c}{m+3}\right)}{\hat{q}_1'\left(t + \frac{c}{m+3}\right)} \in \mathbb{R}$.
But (42) cannot hold: if it did, it would imply that $I, \hat{B}_r, \cdots, \hat{B}_r^{l+1}$ were dependent, which contradicts the fact that the minimal polynomial of $\hat{B}_r$ has degree $l+2$. We conclude that P_1 cannot have multiple roots. We will now show that the only possible constant root of P_1 is $\lambda + \frac{(m+2)c}{m+3}$ and it is a root if and only if $\lambda \neq 0$ and $\alpha = \mu_j = 0$ for any j. For this, let t be a constant root of P_1. Taking the derivative with respect to r of the equality $P_1(t) = 0$ and using Lemma 16, we get

$$\left(t - \frac{(m+2)c}{m+3} - \lambda\right) \hat{q}_1\left(t + \frac{c}{m+3}\right) = 0,$$

from where we deduce that $t = \lambda + \frac{(m+2)c}{m+3}$, since $\hat{q}_1\left(t + \frac{c}{m+3}\right) \neq 0$. Moreover, $P_1(t) = 0$ if and only if

$$\lambda \hat{p}_r(\gamma) + \frac{f}{r^2}\hat{q}_1(\gamma) = 0. \tag{43}$$

Equality (43) forces $\lambda \neq 0$ (if $\lambda = 0$, then, from (43), $\hat{q}_1(\gamma) = 0$; also, $c = \gamma$; recall however that we are under the hypothesis $\hat{q}_1(c) \neq 0$; we obtain a contradiction). Therefore, $\lambda \neq 0$ and then $A_r = r^4 e^{\lambda r^2} A$. Relation (43) is equivalent with $\lambda\left(\widetilde{\hat{b}_r}(\gamma)w, w\right) + \hat{q}_1(\gamma)e^{\lambda r^2}(Aw, w) = 0$, for any $w \in W$ null, where $\widetilde{\hat{b}_r}$ denotes the reduced adjoint operator of $\hat{B}_r$. With the notations of Remark 2, $\widetilde{\hat{b}_r}(\gamma)$, as well as A, act trivially on the subspaces W_j (for $j \geq 1$) of W (here and below the Reader is referred to Lemma 3 of [8], which describes the action of the reduced adjoint operator of a k-step parabolic hermitian operator when applied to the parabolic eigenvalue). It follows that (43) is equivalent with

$$\lambda \widetilde{\hat{b}_r}(\gamma) + \hat{q}_1(\gamma)e^{\lambda r^2} A = 0. \tag{44}$$

We claim that equality (44) holds if and only if $\alpha = \mu_j = 0$ for any j (and $\lambda \neq 0$). Notice first that if (44) holds then $\hat{q}_1(\gamma) \neq 0$ (since $\widetilde{\hat{b}_r}(\gamma) \neq 0$), which implies that $\hat{B}_r$ is 1-step parabolic. We deduce that $\mu_j = 0$ for any j. With the notations of Remark 2, $\widetilde{\hat{b}_r}(\gamma)$, as well as A, act trivially on $\widehat{W_0}$; on $\mathrm{Span}\{e_0, e_1\}$, $\widetilde{\hat{b}_r}(\gamma)$ acts by $\lambda\left(\alpha - \frac{e^{\lambda r^2}}{\lambda}\right)\prod_{j=1}^{s}(\gamma - \beta_j)A$. Relation (44) is equivalent to $\left(\alpha\lambda - e^{\lambda r^2}\right)\prod_{j=1}^{s}(\gamma - \beta_j) + \hat{q}_1(\gamma)e^{\lambda r^2} = 0$; since $\hat{q}_1(\gamma) = \prod_{j=1}^{s}(\gamma - \beta_j)$ and $\lambda \neq 0$, it reduces to $\alpha = 0$. Since P_1 doesn't have multiple roots, $\lambda + \frac{(m+2)c}{m+3}$ is the (unique) simple root of P_1.

Now we consider the case when $\hat{q}_1(c) = 0$. Clearly, $\frac{(m+2)c}{m+3}$ is a root in this case. Define the polynomial $\hat{q}_2(t) := \frac{\hat{q}_1(t)}{t-c}$ and let t be a constant root of

$$\hat{p}_r\left(t + \frac{c}{m+3}\right) + \frac{f}{r^2}\hat{q}_2\left(t + \frac{c}{m+3}\right) = 0. \tag{45}$$

Taking the derivative with respect to r of (45) and using Lemma 16, we get

$$\left(t - \frac{(m+2)c}{m+3} - \lambda\right)\hat{q}_2\left(t + \frac{c}{m+3}\right) = 0.$$

This implies that $t = \lambda + \frac{(m+2)c}{m+3}$, because $\hat{q}_2\left(t + \frac{c}{m+3}\right) \neq 0$ (because $\hat{p}_r$ has no constant roots). As before, t is a root of (45) if and only if

$$\widetilde{\hat{b}_r}(\gamma) + \hat{q}_2(\gamma)e^{\lambda r^2} A = 0. \tag{46}$$

Notice first that if (46) holds, then $\hat{q}_2(\gamma) \neq 0$. Next, we prove that if (46) holds then $\hat{B}_r$ is 1-step parabolic. The argument is the following: suppose, on the contrary, that (46) holds and that $\hat{B}_r$ is 2-step parabolic; then $\lambda = 0$; otherwise, since $(t - \gamma)$ divides $\hat{q}_1$ ($\hat{B}_r$ being 2-step parabolic), $\hat{q}_2(\gamma) = 0$, which is impossible. On the other hand, $\widetilde{\hat{b}_r}(\gamma)$ acts as $\sum_{k=2}^{n}|\mu_k|^2\prod_{j=1}^{s}(\gamma - \beta_j)A$ on $\mathrm{Span}\{e_0, e_1\}$, when $\hat{B}_r$ is 2-step

parabolic; also $\hat{q}_2(\gamma)=\prod_{j=1}^{s}(\gamma-\beta_j)$, when $\lambda=0$; from (46) it follows that

$$\left(\sum_{k=2}^{n}|\mu_k|^2+1\right)\prod_{j=1}^{s}(\gamma-\beta_j)A=0$$

on Span$\{e_0,e_1\}$, which cannot hold. We have proved that if $\lambda+\frac{(m+2)c}{m+3}$ is a root, then $\hat{B}_r$ is 1-step parabolic. Moreover, in this case $\lambda\neq 0$ (if $\lambda=0$ then $c=\gamma$ and since $\hat{q}_1(c)=0$, then $\hat{q}_1(\gamma)=0$ which is absurd because $(t-\gamma)^3$ does not divide the minimal polynomial $\hat{q}$ of $\hat{B}_r$ when $\hat{B}_r$ is 1-step parabolic). Finally, when $\hat{B}_r$ is 1-step parabolic and $\lambda\neq 0$, relation (46) becomes

$$\left(\alpha-\frac{e^{\lambda r^2}}{\lambda}\right)\prod_{j=1}^{s}(\gamma-\beta_j)+\hat{q}_2(\gamma)e^{\lambda r^2}=0$$

which holds if and only if $\alpha=0$, because

$$\hat{q}_2(\gamma)=\frac{1}{\gamma-c}\hat{q}_1(\gamma)=\frac{1}{\lambda}\hat{q}_1(\gamma)=\frac{1}{\lambda}\prod_{j=1}^{s}(\gamma-\beta_j).$$

We have proved that P_1 has an additional constant root, besides $\frac{(m+2)c}{m+3}$, if and only if $\hat{B}_r$ is 1-step parabolic (i.e. $\mu_j=0$ for any j), $\lambda\neq 0$ and $\alpha=0$. The additional constant root is $\lambda+\frac{(m+2)c}{m+3}$. It is easy to see that it is simple.

□

In Proposition 19 we shall determine the Bryant minimal and characteristic polynomials of (M,ω,J). For the proof of this Proposition we need the following additional Lemma.

Lemma 18. *For any $t\in\mathbb{R}$, $(x,r)\in M$ and $w\in x$ non-zero,*

$$g_{(x,r)}\left(d^H\hat{p}_r(t),d^H\hat{p}_r(t)\right)=4\left(\hat{q}'(t)\hat{p}_{r,x}(t)-\hat{q}(t)\hat{p}'_{r,x}(t)-2t\hat{p}^2_{r,x}(t)+\hat{p}^2_{r,x}(t)\frac{(B_r^2w,w)}{r^2(B_rw,w)}\right)$$

$$g_{(x,r)}\left(d^H\left(\frac{f}{r^2}\right),d^H\left(\frac{f}{r^2}\right)\right)=\frac{4f^2}{r^6}\left(\frac{(B_r^2w,w)}{(B_rw,w)}-2r^2\gamma\right)$$

$$g_{(x,r)}\left(d^H\left(\frac{f}{r^2}\right),d^H\hat{p}_r(t)\right)=\frac{4f}{r^2}\left((t-\gamma)\hat{q}_1(t)-(t+\gamma)\hat{p}_{r,x}(t)+\frac{\hat{p}_{r,x}(t)(B_r^2w,w)}{r^2(B_rw,w)}\right).$$

Proof. The first equality follows from Lemma 3, applied to $\hat{B}_r$. To prove the second equality, we remark that the 1-form $d^H\left(\frac{f}{r^2}\right)$ corresponds to the vector field $\frac{1}{r^4}v$ by means of the metric g. Therefore, $g\left(d^H\left(\frac{f}{r^2}\right),d^H\left(\frac{f}{r^2}\right)\right)$ is equal to $\frac{1}{r^6}g_r(v,v)$. The second equality of the Lemma follows from (34). To prove the third equality we notice that $g\left(d^H\left(\frac{f}{r^2}\right),d^H\hat{p}_r(t)\right)$ is equal to $\frac{\hat{L}_t(f)}{r^2}$, where $\hat{L}_t$ is the vector field on M which, at a point $(x,r)\in M$, belongs to $H_x=\mathrm{Hom}_{\mathbb{C}}(x,W/x)$ and is the endomorphism

$$(\hat{L}_t)_{(x,r)}(w)=2\left(\tilde{\hat{b}}_r(t)w-\hat{p}_r(t)\hat{B}_rw\right)\quad \mathrm{mod} w,\quad w\in x.$$

This is true since $\hat{L}_t$ corresponds to the 1-form $d^H\hat{p}_r(t)$ by means of the metric g (which coincides with $\hat{g}_r$ on the bundle H, restricted to a level set N_r). It is straightforward to check that

$$\begin{aligned}(A_r\hat{L}_t w, w) &= 2\left((t-\gamma)\hat{q}_1(t) - \gamma\hat{p}_{r,x}(t)\right)(A_r w, w)\\ (B_r\hat{L}_t w, w) &= 2\hat{p}_{r,x}(t)\left(t - \frac{(B_r^2 w, w)}{r^2(B_r w, w)}\right)(B_r w, w),\end{aligned}$$

so that

$$\hat{L}_t(f)_{(x,r)} = 4f\left((t-\gamma)\,\hat{q}_1(t) - (t+\gamma)\hat{p}_{r,x}(t) + \frac{\hat{p}_{r,x}(t)(B_r^2 w, w)}{r^2(B_r w, w)}\right),$$

which proves the third equality.

□

Proposition 19. *The Bryant characteristic polynomial of (M, g, J) is equal to*

$$p_c(t) = \left(t - \frac{(m+2)c}{m+3}\right)\left(t + \frac{c}{m+3} - \gamma\right)^{n+1}\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_i\right)^{n_j}$$

The Bryant minimal polynomial p_m of (M, g, J) has the following expression:

(1) *if there is $\mu_j \neq 0$ and c is different from β_k (for any k) and γ, then*

$$p_m(t) = \left(t - \frac{(m+2)c}{m+3}\right)\left(t + \frac{c}{m+3} - \gamma\right)^3\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right).$$

(2) *if there is $\mu_j \neq 0$ and c is equal to β_j (for a certain j) or to γ, then*

$$p_m(t) = \left(t + \frac{c}{m+3} - \gamma\right)^3\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right).$$

(3) *if all $\mu_j = 0$ and c is different from β_j (for any j) then*

$$p_m(t) = \left(t - \frac{(m+2)c}{m+3}\right)\left(t + \frac{c}{m+3} - \gamma\right)^2\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right).$$

except when $\lambda \neq 0$ and $\alpha = 0$, in which case

$$p_m(t) = \left(t - \frac{(m+2)c}{m+3}\right)\left(t + \frac{c}{m+3} - \gamma\right)\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right).$$

(4) *if all $\mu_j = 0$ and c is equal to β_j (for a certain j), then*

$$p_m(t) = \left(t + \frac{c}{m+3} - \gamma\right)^2\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right)$$

except when $\alpha = 0$ (and $\lambda \neq 0$), when

$$p_m(t) = \left(t + \frac{c}{m+3} - \gamma\right)\prod_{j=1}^{s}\left(t + \frac{c}{m+3} - \beta_j\right).$$

Proof. Let t be a non-constant root of the polynomial P_1. Then

$$\left(t-\frac{(m+2)c}{m+3}\right)\hat{p}_r\left(t+\frac{c}{m+3}\right)=-\frac{f}{r^2}\hat{q}_1\left(t+\frac{c}{m+3}\right)$$

$$\begin{aligned}\left(t-\frac{(m+2)c}{m+3}\right)^2\hat{p}_r'\left(t+\frac{c}{m+3}\right)&=\left(t-\frac{(m+2)c}{m+3}\right)P_1'(t)+\frac{f}{r^2}\hat{q}_1\left(t+\frac{c}{m+3}\right)\\&-\frac{f}{r^2}\left(t-\frac{(m+2)c}{m+3}\right)\hat{q}_1'\left(t+\frac{c}{m+3}\right).\end{aligned}$$

Using these relations and Lemma 18, we can calculate the square norm of the covector $(d^H P_1)(t)$ as follows:

$$\begin{aligned}g\left((d^HP_1)(t),(d^HP_1)(t)\right)&=-4\hat{q}\left(t+\frac{c}{m+3}\right)\left(t-\frac{(m+2)c}{m+3}\right)P_1'(t)\\&-\frac{4f}{r^2}\left(t-\frac{(m+2)c}{m+3}\right)\hat{q}'\left(t+\frac{c}{m+3}\right)\hat{q}_1\left(t+\frac{c}{m+3}\right)\\&-\frac{4f}{r^2}\hat{q}\left(t+\frac{c}{m+3}\right)\hat{q}_1\left(t+\frac{c}{m+3}\right)\\&+\frac{4f}{r^2}\left(t-\frac{(m+2)c}{m+3}\right)\hat{q}\left(t+\frac{c}{m+3}\right)\hat{q}_1'\left(t+\frac{c}{m+3}\right)\\&+\frac{8f}{r^2}\left(t-\frac{(m+2)c}{m+3}\right)\left(t+\frac{c}{m+3}-\gamma\right)\hat{q}_1^2\left(t+\frac{c}{m+3}\right).\end{aligned}$$

Since $\hat{q}'(t)\hat{q}_1(t)-\hat{q}(t)\hat{q}_1'(t)=2(t-\gamma)\hat{q}_1^2(t)$, the above expression reduces to

$$\begin{aligned}g((d^HP_1)(t),(d^HP_1)(t))&=-4\hat{q}\left(t+\frac{c}{m+3}\right)\left(t-\frac{(m+2)c}{m+3}\right)P_1'(t)\\&-\frac{4f}{r^2}\hat{q}_1^2\left(t+\frac{c}{m+3}\right)\left(t-\gamma+\frac{c}{m+3}\right)^2.\end{aligned}$$

On the other hand, Lemma 16 together with the definition of P_1 imply that

$$\left(\frac{d}{dr}P_1\right)(t)=-\frac{2f}{r}\hat{q}_1\left(t+\frac{c}{m+3}\right)\left(t-\frac{(m+2)c}{(m+3)}-\lambda\right).$$

Since $g(dr,dr)=\frac{1}{f}$, we obtain

$$\begin{aligned}g\left((dP_1)(t),(dP_1)(t)\right)&=g\left((d^HP_1)(t),(d^HP_1)(t)\right)+\left(\frac{d}{dr}P_1(t)\right)^2g\left(dr,dr\right)\\&=-4\hat{q}\left(t+\frac{c}{m+3}\right)\left(t-\frac{(m+2)c}{m+3}\right)P_1'(t).\end{aligned}$$

On the other hand, since $P_1'(t)dt+d\left(P_1(t)\right)=0$, we get

$$g(dt,dt)=\frac{g\left((dP_1)(t),(dP_1)(t)\right)}{P_1'(t)^2}. \tag{47}$$

We distinguish three cases: (i) P_1 has no constant roots; (ii) P_1 has a unique constant root, which is simple and equal to $t_1=\lambda+\frac{(m+2)c}{m+3}$; (iii) P_1 has two constant roots, t_1 and $t_2=\frac{(m+2)c}{m+3}$, which are simple and distinct (see Proposition

17). In all cases, the Bryant characteristic polynomial of (M,ω,J) is

$$p_c(t) = \left(t - \frac{(m+2)c}{m+3}\right)\hat{Q}\left(t + \frac{c}{m+3}\right).$$

In case (i), the Bryant minimal polynomial of (M,g,J) is

$$p_m(t) = \hat{q}\left(t + \frac{c}{m+3}\right)\left(t - \frac{(m+2)c}{m+3}\right)$$

In case (ii),

$$p_m(t) = (t-t_1)^{-1}\left(t - \frac{(m+2)c}{m+3}\right)\hat{q}\left(t + \frac{c}{m+3}\right).$$

and in case (iii),

$$p_m(t) = (t-t_1)^{-1}(t-t_2)^{-1}\left(t - \frac{(m+2)c}{m+3}\right)\hat{q}\left(t + \frac{c}{m+3}\right).$$

$\square$

Proof of Proposition 13: Proposition 13 is an easy consquence Proposition 19.

5. Examples

In this section we consider some important classes of Bochner-flat Kähler manifolds and we show how they can be realised locally as generalised Kähler cones.

(i) *Bryant's Bochner-flat Kähler structures.* Let $N = S^{2m+1} \subset \mathbb{C}^{m+1}$ with its standard CR structure and $(k_1,\cdots,k_{m+1})$ a system of non-negative real numbers. Define, for every $r>0$, the vector field

$$T_r(z) := \sum_{j=1}^{m+1}\left(1+k_j r^2\right)\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right),$$

which is the Reeb vector field of a Sasaki structure on $S^{2m+1} \subset \mathbb{C}^{m+1}$. Here $z=(z_1,\cdots,z_{m+1})$ belongs to S^{2m+1}, $z_j = x_j + iy_j$ for any $j \in \{1,\cdots,m+1\}$ and $r^2 = |z_1|^2+\cdots+|z_{m+1}|^2$. The family of Sasaki Reeb vector fields $\{T_r, r>0\}$ defines a Bochner-flat generalised Kähler cone on $\mathbb{C}^{m+1}\setminus\{0\}$, which belongs to the first case of Proposition 11; the solution of equation (19) is $\mu(t) = -\frac{2}{t}$ and the hermitian operator B is semi-simple, with eigenvalues $k_j' = k_j - \frac{1}{m+2}\sum_{i=1}^{m+1}k_i$, for $j \in \{1,\cdots,m+1\}$. This Bochner-flat Kähler structure has been discovered by Robert Bryant in [3] and has been further studied in [8]; it can be extended as a complete Bochner-flat Kähler structure on $\mathbb{C}^{m+1}$.

(ii) *Bochner-flat Kähler-Einstein structures.* Let (M,g,J) be a Bochner-flat generalised Kähler cone. With the notations of Proposition 11, suppose that $B = eA$, for $e \in \mathbb{R}$. If (M,g,J) belongs to the first and second case of Proposition 11, then it is Kähler-Einstein if and only if $e^2+2d=0$; moreover, the Bryant modified Ricci operator of (M,ω,J) is $\Theta = \frac{e}{m+3}\mathrm{Id}$ in the first case and $\Theta = -\frac{e}{m+3}\mathrm{Id}$ in the second case. If (M,g,J) belongs to the third case of Proposition 11 and $B=eA$ then it is never Einstein; if it belongs to the fourth case of Proposition 11 then it is Kähler-Einstein if and only if $e=0$ (and $\lambda<0$); The Bryant modified Ricci

operator is $\Theta = \frac{\lambda}{m+3}\mathrm{Id}$ in this case.

(iii) *Bochner-flat generalised Kähler cones of order one.* If $B = eA$ but (M, g, J) is not Einstein, then it must have order one. The Bryant minimal and characteristic polynomials have the following expressions: if (M, g, J) belongs to the case 1, respectively to the case 2 of Proposition 11, then

$$p_c(t) = \left(t \mp \frac{e}{m+3}\right)^{m+1} \left(\left(t \mp \frac{e}{m+3}\right)^2 \pm e\left(t \mp \frac{e}{m+3}\right) + \frac{e^2+2d}{4}\right)$$
$$p_m(t) = \left(t \mp \frac{e}{m+3}\right) \left(\left(t \mp \frac{e}{m+3}\right)^2 \pm e\left(t \mp \frac{e}{m+3}\right) + \frac{e^2+2d}{4}\right);$$

(M, g, J) is of hyperbolic type when $d > 0$, of 1-step parabolic type when $d = 0$ and of elliptic type when $d < 0$. If (M, J, g) belongs to the cases 3 and 4 of Proposition 11, then

$$p_c(t) = \left(t + \frac{(m+2)\lambda}{m+3}\right)\left(t - \frac{\lambda}{m+3}\right)^{m+2}$$
$$p_m(t) = \left(t + \frac{(m+2)\lambda}{m+3}\right)\left(t - \frac{\lambda}{m+3}\right)^{2}.$$

(Recall that $\lambda = 0$ in the case 3 and $\lambda \neq 0$ in case 4). When (M, ω, J) belongs to the case 3, it is 2-step parabolic; when it belongs to case 4, it is 1-step parabolic. Bochner-flat Kähler structures of order one have been studied in [1]. As shown in [1], [8], they fiber over a Kähler manifold with constant holomorphic sectional curvature (in our formalism, the fibration is $M \to N_r/\tilde{T}_r$).

(iv) *Weighted projective spaces as generalised Kähler cones.* Let $\mathbb{P}^{m+1}_a$ be a weighted projective space, of weights $(a_1, \cdots, a_{m+1})$, where a_j are positive integers. As shown in [3], [8], $\mathbb{P}^{m+1}_a$ has a canonical Bochner-flat Kähler structure, of semi-simple type, isomorphic with M_C, where C is a hermitian semi-simple operator of $\mathbb{C}^{m+2,1}$, with eigenvalues $-\sum_{j=1}^{m+2}\lambda_j, \lambda_1, \cdots, \lambda_{m+2}$, where λ_j are related to the weights a_j by $\lambda_j = a_j - \frac{1}{m+3}\sum_{i=1}^{m+2} a_i$, for any $j \in \{1, \cdots, m+2\}$. As a Bochner-flat generalised Kähler cone, $\mathbb{P}^{m+1}_a$ belongs to the first case of Proposition 11; μ is any solution of equation (19), with $d = \frac{2a^2_{m+2}}{(m+3)^2}$, and the hermitian operator B is semi-simple, with eigenvalues $-\frac{\sum_{j=1}^{m+1} a_j}{m+2}, a_1 - \frac{\sum_{j=1}^{m+1} a_j}{m+2}, \cdots, a_{m+1} - \frac{\sum_{j=1}^{m+1} a_j}{m+2}$.

(v) *Tachibana and Liu Bochner-flat Kähler generalised Kähler cones.* Consider a Bochner-flat generalised Kähler cone structure (g, J) which belongs to the first case of Proposition 11. With the notations of Section 4, assume that $k_1 = \cdots = k_{m+1} := \bar{k}$. On the set $D \subset \mathbb{C}^{m+1}$ defined by (24), the Kähler structure (g, J) has a Kähler potential x which depends only on $r^2 := |z_1|^2 + \cdots + |z_{m+1}|^2$. As a function of $r^2 = t$, x satisfies the implicit equation

$$e^{ax(t)}\dot{\mu}(x(t))^{-\frac{1}{2}} = t, \tag{48}$$

where $a := (m+2)\bar{k}$. In general, a Kähler structure which is defined on an open subset of the standard $\mathbb{C}^m$ and has a global Kähler potential, say h, which depends

only on r^2 is Bochner-flat, if and only if h, as a function of $r^2 = t$, satisfies a differential equation of the form [14]

$$\ddot{h}(t) = \lambda_1 t\dot{h}^3(t) + \lambda_2 \dot{h}^2(t), \tag{49}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. It can be easily verified that if x satisfies (48), then it satisfies also (49), with $\lambda_1 := a^2 + \frac{d}{2}$ and $\lambda_2 := -2a$.

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